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LETTER TO THE EDITOR

Superconformal invariance in the Ashkin–Teller quantum chain with free boundary conditions

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Abstract. The finite-size limit of the lower part of the spectrum of the Ashkin-Teller chain with free boundary conditions is studied numerically and interpreted from the point of view of conformal invariance. Several irreducible representations of the Virasoro algebra with the central charge $c = 1$ are identified. For two special values of the coupling constant, higher degeneracies occur and the whole spectrum can be understood in terms of a few irreducible representations of the $N = 2$ superconformal algebra.

In a previous paper (von Gehlen and Rittenberg 1968a) we have have studied the lowest part of the spectrum of the four-state Ashkin-Teller (1943) quantum chain introduced by Kohmoto *et al* (1981) and we have found a few critical exponents. In the present letter we show some new results which have been obtained from the study of higher excitations. The model is defined by the Hamiltonian

$$H = -\frac{1}{2(1+\varepsilon)} \sum_{i=1}^{N} \{[\sigma_i + \sigma_i^+ + \varepsilon(\sigma_i)^2] + \lambda[\Gamma_i\Gamma_{i+1}^+ + \Gamma_i^+\Gamma_{i+1} + \varepsilon(\Gamma_i)^2(\Gamma_{i+1})^2]\} \tag{1}$$

where N represents the number of sites, λ playes the role of the inverse of the temperature, ε is a coupling constant and

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2}$$

The model is self-dual and has a continuous phase transition at $\lambda = 1$ characterised by a central charge of the Virasoro algebra $c = 1$ in the region $-1 < \varepsilon \leqslant 1$. (Actually the phase diagram shows a richer structure but in this letter we confine ourselves to study only the line $\lambda = 1$, $-1 < \varepsilon \leqslant 1$.) Since for $c = 1$, in the absence of higher symmetries, the critical exponents are not constrained as in the case $c < 1$ to a finite set of rational numbers (Friedan *et al* 1984), we can expect that for each value of ε one can find an infinite number of unrelated critical exponents and that for various values of ε one finds other exponents. The situation could change, however, if higher symmetries are present which would again discretise the critical exponents. It is the aim of this letter to clarify this point.

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It was noticed by Kohmoto *et al* (1981) that for the considered interval of ε it is convenient to consider the function $x_T(\varepsilon)$:

$$x_T = \frac{\pi}{2\cos^{-1}(-\varepsilon)} \qquad (\infty > x_T \geq \tfrac{1}{2}) \tag{3}$$

in terms of which the critical exponents have simple expressions ($x_T(\varepsilon)$ is the scaling dimension of the energy density operator which varies continuously with ε). Some special points are already known: $x_T = \frac{1}{2}$ corresponds to the four-state Potts model, $x_T = 1$ describes two uncoupled Ising models and $x_T = 2$ is a Kosterlitz–Thouless point. Another special point is $x_T = \frac{2}{3}$ which has a higher symmetry as discussed by Zamolodchikov and Fateev (1985). As will be shown in this letter there are two other special points $x_T = \frac{4}{3}$ and 3 where one has superconformal invariance.

We consider the Hamiltonian (1) with free boundary conditions ($\Gamma_{N+1} = 0$). The Hamiltonian commutes with the Z_4 charge operator:

$$\hat{Q} = \sum_{i=1}^{N} q_i \pmod 4 \tag{4a}$$

where

$$q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \tag{4b}$$

Because of charge conservation, the Hamiltonian splits into four charge sectors and we denote the corresponding matrices by $H_Q^{(F)}$ ($Q = 0, 1, 2$ and 3). From the invariance under charge conjugation we have

$$H_1^{(F)} = H_3^{(F)} \tag{5}$$

and we will thus study only the matrices with $Q = 0$, 1 and 2. Let $E_Q^{(F)}(s)(s = 0, 1, \ldots)$ be the energy levels in the charge Q sector for N sites. We consider the quantities (Cardy 1984, 1986, von Gehlen and Rittenberg 1986b)

$$\mathscr{E}_Q(s) = \lim_{N\to\infty} \frac{N}{\pi\xi}\left(E_Q^{(F)}(s) - E^{(F)}\right) \tag{6}$$

where $E^{(F)} = E_0^{(F)}(0)$ is the ground state of the system. The constant ξ gives the Euclidean timescale and can be determined using the methods described in von Gehlen and Rittenberg (1986a). ξ is, of course, a function of ε. It is a consequence of conformal invariance in two dimensions that the quantities $\mathscr{E}_Q(s)$ are described by unitary irreducible representations (IR) of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{m+n} + \tfrac{1}{12}c(n^3 - n)\delta_{n,-m} \qquad n, m \in Z \tag{7}$$

namely, an IR of the Virasoro algebra characterised by the highest weight Δ:

$$L_0|\Delta\rangle = \Delta|\Delta\rangle \qquad L_n|\Delta\rangle = 0 \qquad (n \geq 1) \tag{8}$$

gives a contribution

$$\mathscr{E}_Q(r) = \Delta + r \qquad (r = 0, 1, 2, \ldots) \tag{9}$$

to the spectrum $\mathscr{E}_Q(s)$. Δ is a surface critical exponent (Binder 1983, Cardy 1984) and the level $\Delta+r$ has a degeracy $D(\Delta, r)$ which for $c<1$ can be obtained from the character formulae of Rocha-Caridi (1985). For $c=1$, which is our case, the function $D(\Delta, r)$ is independent of Δ and equal to the function $\pi(r)$ determined by the partition function

$$\prod_{n=1}^{\infty} (1-q^n)^{-1} = \sum_{r=0}^{\infty} \pi(r) q^r \tag{10}$$

unless $\Delta = \frac{1}{4}t^2$ where t is an integer. In the latter case $D(\frac{1}{4}t^2, r)$ can be determined from the character formulae of Kac (1979).

From the measured spectrum $\mathscr{E}_Q(s)$ one can determine which IR (Δ) contribute to the spectrum and thus the operator content of the Hamiltonian, i.e. the surface critical exponents of the system. One more observation is in order. As a consequence of the parity invariance of the Hamiltonian (1) the spectrum $\mathscr{E}_Q(s)$ is composed by the parity even ($\mathscr{E}_Q^{(+)}$) and odd ($\mathscr{E}_Q^{(-)}$) sectors:

$$\mathscr{E}_Q = \mathscr{E}_Q^{(+)} + \mathscr{E}_Q^{(-)} \tag{11}$$

If the IR (Δ) has the highest weight in the sector $\mathscr{E}_Q^{(P)}$ $(P=\pm)$, we will denote the IR by $(\Delta)^P$. The level $|\Delta+r\rangle$ has the relative parity $(-1)^r$ to the level $|\Delta\rangle$. This is a consequence of the invariance under the transformation

$$L_n \to (-1)^n L_n \qquad (n \in Z) \tag{12}$$

of the Virasoro algebra.

Without going into details, we have checked numerically for several values of ε that the $\mathscr{E}_Q$ contain the following IR with the right degeneracies:

$$\begin{aligned}
\mathscr{E}_0 &= (0)^+ \oplus \left(\frac{2}{x_T}\right)^- \oplus \left(\frac{8}{x_T}\right)^+ \oplus \cdots \\
\mathscr{E}_1 = \mathscr{E}_3 &= \left(\frac{1}{2x_T}\right)^+ \oplus \left(\frac{9}{2x_T}\right)^+ \oplus \cdots \\
\mathscr{E}_2 &= (1)^+ \oplus \left(\frac{2}{x_T}\right)^+ \oplus \cdots .
\end{aligned} \tag{13}$$

The IR given in (13) do not exhaust the operator content of the spectrum but they describe its lowest part. A closer examination of the x_T dependence of the surface exponents given by (13) as well as of the bulk exponents (von Gehlen and Rittenberg 1986a) suggest that the $x_T = \frac{4}{3}$ and $x_T = 3$ are points of higher symmetry. This guess was confirmed by our study as will be shown below.

We first present in detail the $x_T = \frac{4}{3}$ case. Chains from 2-10 sites have been considered and the Van den Broeck-Schwartz (1979) approximants for the $\mathscr{E}_Q^{(\pm)}$ are presented in tables 1-3. (The normalisation factor is $\pi\xi = 3.7614(3)$). A cursory look at the spectra would suggest the following operator content of the Hamiltonian:

$$\begin{aligned}
\mathscr{E}_0 &= (0)^+ \oplus (\tfrac{3}{2})^+ \\
\mathscr{E}_1 = \mathscr{E}_3 &= (\tfrac{3}{8})^+ \\
\mathscr{E}_2 &= (1)^+ \oplus (\tfrac{3}{2})^+ .
\end{aligned} \tag{14}$$

This picture is not correct, however, since it does not give the right degeneracies. For the level 4 ($\mathscr{E}_0^{(+)}$) one finds three levels instead of two and for the level $\frac{3}{8}+3$ one finds

Table 1. The finite-size limit spectrum $\mathscr{E}_0$ in the charge zero sector for $x_T = \frac{4}{3}$. The degeneracies D are computed considering the IR (0) and ($\frac{3}{2}$) of the Virasoro algebra with $c = 1$. The degeneracies d are computed considering the IR $(0)_1$ of the $N = 1$ superconformal algebra. The levels $a, b, \ldots, f$ are exactly degenerate with the corresponding levels of table 3 for any number of sites.

$0+\frac{1}{2}r$	d	D	$\mathscr{E}_0^{(+)}$	$\mathscr{E}_0^{(-)}$
0	1	1	0	—
$\frac{3}{2}$	1	1	—	1.4999 (1)[a]
2	1	1	2.000	—
$\frac{5}{2}$	1	1	2.500 (1)[b]	—
3	1	1	—	3.0000 (7)
$\frac{7}{2}$	2	2	—	3.499 (3)[c], 3.5000 (6)[d]
4	3	2	4.001 (3), 4.0000 (6), 4.0001 (5)	—
$\frac{9}{2}$	3	3	4.49 (2)[e], 4.49 (1)[f], 4.500 (1)	—
5	3	2	—	4.993 (8), 4.97 (7), 5.01 (3)
$\frac{11}{2}$	5	5	—	5.3 (2), 5.49 (1)

Table 2. The finite-size limit $\mathscr{E}_1$ in the charge one sector for $x_T = \frac{4}{3}$. The degeneracies D are computed considering the IR ($\frac{3}{8}$) of the Virasoro algebra. The degeneracies d are computed considering the IR $(\frac{3}{8})_1$ of the $N = 1$ superconformal algebra.

$\frac{3}{8}+r$	d	D	$\mathscr{E}_1^{(+)}$	$\mathscr{E}_1^{(-)}$
$\frac{3}{8}=0.375$	1	1	0.374 99 (2)	—
$\frac{3}{8}+1$	1	1	—	1.375 00 (6)
$\frac{3}{8}+2$	2	2	2.3749 (5), 2.3750 (3)	—
$\frac{3}{8}+3$	4	3	3.3753 (8)	3.374(4), 3.3749 (5), 3.376 (2)
$\frac{3}{8}+4$	6	5	4.36 (3), 4.374 (5), 4.375 (1) 4.3749 (6), 4.35 (4)	4.376 (7)
$\frac{3}{8}+5$	9	7	—	5.4 (1), 5.3 (2)

Table 3. The finite-size limit $\mathscr{E}_2$ in the charge two sector for $x_T = \frac{4}{3}$. The degeneracies D are computed considering the IR (1) and ($\frac{3}{2}$) of the Virasoro algebra. The degeneracies d are computed considering the IR $(1)_1$ of the $N = 1$ superconformal algebra. The levels $a, b, \ldots, f$ are exactly degenerate with the corresponding levels of table 1.

$1+\frac{1}{2}r$	d	D	$\mathscr{E}_2^{(+)}$	$\mathscr{E}_2^{(-)}$
1	1	1	0.999 98 (6)	—
$\frac{3}{2}$	1	1	1.4999 (1)[a]	—
2	1	1	—	2.0000 (3)
$\frac{5}{2}$	1	1	—	2.500 (1)[b]
3	2	2	3.000 (1) (6), 3.004 (8)	—
$\frac{7}{2}$	2	2	3.499 (3)[c], 3.5000 (6)[d]	—
4	2	2	—	4.000 (1), 3.98 (5)
$\frac{9}{2}$	3	3	—	4.49 (2)[e], 4.49 (1)[f], 4.5 (1)
5	4	4	5.01 (5), 4.96 (3)	—

four levels (one with the wrong parity) instead of three. (The values of the degeneracies D have been computed for us by Altschüler and Lacki (1986).) This mismatch of the degeneracies suggests a higher symmetry of the spectrum. We now recall that for $c=1$ the Virasoro algebra can be enlarged to the $N=1$ and $N=2$ superconformal algebras.

The $N=1$ superconformal algebra contains besides the Virasoro generators L_n, the odd generators G_r (Friedan *et al* 1985, Berdshadski *et al* 1985, Eichenherr 1985). The corresponding superalgebra contains besides (7) the following commutation and anticommutation relations:

$$[L_n, G_r] = (\tfrac{1}{2}m - r)G_{m+r}$$
$$\{G_r, G_s\} = 2L_{r+s} + \tfrac{1}{3}c(r^2 - \tfrac{1}{4})\delta_{r,-s} \qquad (15)$$

where $r, s \in Z$ for the Ramond sector and $r, s \in Z + \frac{1}{2}$ in the Neveu-Schwartz sector. For $c=1$ the possible values of Δ are

$$\begin{array}{ll} \text{Neveu-Schwartz sector} & (0)_1, (\tfrac{1}{16}), (\tfrac{1}{6})_1, (1)_1 \\ \text{Ramond sector} & (\tfrac{1}{24})_1, (\tfrac{1}{16})_1, (\tfrac{3}{8})_1, (\tfrac{9}{16})_1. \end{array} \qquad (16)$$

Here $(\Delta)_1$ denotes an IR with the highest weight Δ of $N=1$ extended supersymmetry. An IR has the spectrum $\Delta + \frac{1}{2}r$ ($r = 0, 1, \ldots$) in the Neveu-Schwartz sector and $\Delta + r$ in the Ramond sector. The degeneracies $d(\Delta, r)$ can be derived from the character formulae of Goddard *et al* (1986). The values of $d(\Delta, r)$ for our cases are shown in tables 1-3 and now we notice that we get the correct values of the degeneracies. We thus have instead of (14)

$$\mathscr{E}_0 = (0)_1 \qquad \mathscr{E}_1 = \mathscr{E}_3 = (\tfrac{3}{8})_1 \qquad \mathscr{E}_2 = (1)_1. \qquad (17)$$

That this picture is correct can be also checked from the associative algebra related to the short-distance expansion (Belavin *et al* 1984, Berdhadski *et al* 1985):

$$(0)_1 \otimes (0)_1 = (0)_1 \qquad (0)_1 \otimes (1)_1 = (1)_1 \qquad (1)_1 \otimes (1)_1 = (0)_1 \qquad (18)$$

in agreement with the Z_2 grading of $\mathscr{E}_0$ and $\mathscr{E}_2$.

We now notice that several levels from the $\mathscr{E}_0$ sector are exactly degenerate (for any number of sites) with levels in the $\mathscr{E}_2$ sector (see tables 1 and 3). This suggests a still higher symmetry if we put the $\mathscr{E}_0$ and $\mathscr{E}_2$ sectors together. Let us consider the $N=2$ superconformal algebra (Di Vecchia *et al* 1985, 1986, Waterson 1986, Boucher *et al* 1986). For $c=1$ the IR are

$$\begin{array}{ll} \text{Neveu-Schwartz sector} & (0; 0)_2, (\tfrac{1}{6}; \pm\tfrac{1}{3})_2 \\ \text{Ramond sector} & (\tfrac{3}{8}; 0)_2, (\tfrac{1}{24}; \pm\tfrac{1}{3})_2, (\tfrac{1}{24}; \pm\tfrac{2}{3})_2. \end{array} \qquad (19)$$

Here $(\Delta; q)_2$ represents the IR of the $N=2$ superconformal algebra with highest weight Δ and charge q. (There is a U(1) charge which appears in this case.) The $N=2$ IR can be decomposed in terms of $N=1$ IR. We have

$$\begin{array}{ll} (0; 0)_2 = (0)_1 \oplus (1)_1 & (\tfrac{1}{6}, q)_2 = (\tfrac{1}{6})_1 \\ (\tfrac{3}{8}; 0)_2 = (\tfrac{3}{8})_1 & (\tfrac{1}{24}; q)_2 = (\tfrac{1}{24})_1. \end{array} \qquad (20)$$

With this final observation we can sum up our result. The finite-size limit of the spectrum of the Hamiltonian at $x_T = \frac{4}{3}$ has $N=2$ supersymmetry with the following content of IR:

$$\mathscr{E}_0 + \mathscr{E}_2 = (0; 0)_2 \qquad \mathscr{E}_1 = \mathscr{E}_3 = (\tfrac{3}{8}; 0)_2. \qquad (21)$$

Table 4. The finite-size limit $\mathscr{E}_0$ in the charge zero sector for $x_T = 3$. The degeneracies D are computed using the IR (0) and ($\frac{2}{3}$) of the Virasoro algebra. The degeneracies d are computed using the IR $(0)_1$ and $(\frac{1}{6})_1$ of the $N=1$ superconformal algebra. The levels $a, b, \ldots, f$ are exactly degenerate with the corresponding levels of table 6 for any number of sites.

$0+r$	$\frac{2}{3}+r$	d	D	$\mathscr{E}_0^{(+)}$	$\mathscr{E}_0^{(-)}$
0	—	1	1	0	
—	$\frac{2}{3}$	1	1	—	0.666 (2)
—	$\frac{2}{3}+1$	1	1	1.666 (1)[a]	—
2	—	1	1	2.00	—
—	$\frac{2}{3}+2$	3	2	2.670 (5)[b]	2.670 (5)[b], 2.73 (2)[c]
3	—	1	1	—	3.05 (1)
—	$\frac{2}{3}+3$	4	3	3.66 (3)[d], 3.68 (1)[e]	3.66 (3)[d]
4	—	3	2	4.1 (2), 3.97 (2)	—
—	$\frac{2}{3}+4$	6	5	—	4.63 (3)[f], 4.66 (1)
5		3	2	—	5.00 (1)

Table 5. The finite-size limit $\mathscr{E}_1$ in the charge one sector for $x_T = 3$. The degeneracies D are computed using the IR ($\frac{1}{6}$) and ($\frac{3}{2}$) of the Virasoro algebra. The degeneracies d are computed using the IR $(\frac{1}{6})_1$ and $(0)_1$ (or $(1)_1$) of the superconformal algebra.

$\frac{1}{6}+r$	$\frac{3}{2}+r$	d	D	$\mathscr{E}_1^{(+)}$	$\mathscr{E}_1^{(-)}$
$\frac{1}{6} \approx 0.1666\ldots$	—	1	1	0.1665 (3)	—
$\frac{1}{6}+1$	—	1	1	—	1.1664 (6)
—	$\frac{3}{2}$	1	1	1.497 (4)	—
$\frac{1}{6}+2$	—	2	2	2.165 (1), 2.25 (2)	—
—	$\frac{5}{2}$	1	1	—	2.5 (1)
$\frac{1}{6}+3$	—	3	3	—	3.16 (1), 3.22 (1), 3.3 (2)
—	$\frac{7}{2}$	2	2	3.47 (2), 3.52 (9)	—
$\frac{1}{6}+4$	—	6	5	4.16 (2), 4.17 (1), 4.2 (2)	—
—	$\frac{9}{2}$	3	3	—	4.46 (3), 4.3 (2)

Table 6. The finite-size limit $\mathscr{E}_2$ in the charge two sector for $x_T = 3$. The degeneracies D are computed using the IR ($\frac{2}{3}$) and (1) of the Virasoro algebra. The degeneracies d are computed using IR $(\frac{1}{6})_1$ and $(1)_1$ of the $N=1$ superconformal algebra. The levels $a, b, \ldots, f$ are exactly degenerate with the corresponding levels of table 4.

$\frac{2}{3}+r$	$1+r$	d	D	$\mathscr{E}_2^{(+)}$	$\mathscr{E}_2^{(-)}$
$\frac{2}{3}$	—	1	1	0.666 (2)	—
—	1	1	1	1.003 (8)	—
$\frac{2}{3}+1$	—	1	1	—	1.666 (1)[a]
—	2	1	1	—	2.07 (2)
$\frac{2}{3}+2$	—	3	2	2.670 (5)[b], 2.73 (2)[c]	2.670 (5)[b]
—	3	2	2	2.991 (1), 3.2 (1)	—
$\frac{2}{3}+3$	—	4	3	3.66 (3)[d]	3.66 (3)[d], 3.68 (1)[e]
—	4	2	2	—	3.99 (2)
$\frac{2}{3}+4$	—	6	5	4.63 (3)[f]	—

We now consider the $x_T = 3$ case. The numerical results are shown in tables 4-6. (The normalisation is $\pi\xi = 6.986(5)$). Their interpretation is the following one. The $\mathscr{E}_Q$ can be expressed in terms of the IR of the Virasoro algebra:

$$\mathscr{E}_0 = (0) \oplus (\tfrac{2}{3}) \qquad \mathscr{E}_1 = \mathscr{E}_3 = (\tfrac{1}{6}) \oplus (\tfrac{3}{2}) \qquad \mathscr{E}_2 = (\tfrac{2}{3}) \oplus (1) \tag{22}$$

but then the degeneracies are wrong. We now combine $\mathscr{E}_0$ with $\mathscr{E}_1$ and $\mathscr{E}_2$ with $\mathscr{E}_3$ and describe the spectra in terms of $N = 1$ superconformal IR:

$$\mathscr{E}_0 + \mathscr{E}_1 = (0)_1 \oplus (\tfrac{1}{6})_1 \qquad \mathscr{E}_2 + \mathscr{E}_3 = (\tfrac{1}{6})_1 \oplus (1)_1. \tag{23}$$

In this way we get the correct degeneracies.

The whole spectrum can now be combined in order to obtain $N = 2$ superconformal invariance:

$$\mathscr{E}_0 + \mathscr{E}_1 + \mathscr{E}_2 + \mathscr{E}_3 = (0; 0)_2 \oplus (\tfrac{1}{6}; q)_2 \oplus (\tfrac{1}{6}; q')_2. \tag{24}$$

The two U(1) charges q and q' are unknown to us.

This concludes our observations about the spectra with free boundary conditions. The corresponding results for periodic and twisted boundary conditions will be published elsewhere. One important conclusion of that study is that for periodic boundary conditions and any x_T the value $\Delta = \frac{1}{16}$ appears, which does not figure on the list (19) of the possible values of the highest weight for $N = 2$ superconformal algebra and thus the Hamiltonian with periodic boundary conditions is not $N = 2$ supersymmetric although the Hamiltonian with free boundary conditions is.

Note added in proof. After this letter was submitted for publication, we found another supersymmetric point at $x_T = 12$ with the following IR content:

$$\mathscr{E}_0 = (0)_1 \oplus (\tfrac{1}{6})_1 \qquad \mathscr{E}_1 = \mathscr{E}_3 = (\tfrac{1}{24})_1 \oplus (\tfrac{3}{8})_1 \qquad \mathscr{E}_2 = (1)_1 + (\tfrac{1}{6})_1.$$

The combination of the spectra leads again to $N = 2$ supersymmetry.

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